example

Find the points on the sphere $x^2+y^2+z^2=4$ closest to & farthest from $(3,-1,1)$

optimize distance, subject to sphere

we'll optimize distance squared

$$\begin{cases} \text{optimize: } f(x,y,z) = (x-3)^2+(y+1)^2+(z-1)^2. \\ \text{subject to: } x^2+y^2+z^2=4 \end{cases}$$

$$\begin{cases} \text{OPT: } (x^2-6x+9)+(y^2+2y+1)+(z^2-2z+1) \\ \text{SUB to: } x^2+y^2+z^2=4 \end{cases}$$

$$\begin{cases} \text{OPT: } (x^2+y^2+z^2)+(9+1+1)+(-6x+2y-2z) \\ \text{SUB: } x^2+y^2+z^2-4=0 \end{cases}$$

$$\begin{cases} \text{OPT: } f(x,y,z)=15-6x+2y-2z \\ \text{SUB: } g(x,y,z)=0 \quad g(x,y,z)=x^2+y^2+z^2-4 \end{cases}$$

$$F(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z) = 15-6x+2y-2z-\lambda(x^2+y^2+z^2-4)$$

$$\nabla F = \langle -6-2\lambda x,\ 2-2\lambda y,\ -2-2\lambda z,\ -(x^2+y^2+z^2-4)\rangle$$

$$\nabla F = \vec{0} \quad \text{iff} \quad \begin{cases} -6-2\lambda x=0 \\ 2-2\lambda y=0 \\ -2-2\lambda z=0 \\ -(x^2+y^2+z^2-4)=0 \end{cases} \quad \text{iff} \quad \begin{cases} \lambda x=-3 & (1) \\ \lambda y=1 & (2) \\ \lambda z=-1 & (3) \\ x^2+y^2+z^2=4 & (4) \end{cases}$$

$\lambda \neq 0$ by (1)

multiply (4) by λ^2

$$\lambda^2(x^2+y^2+z^2)=4\lambda^2 \rightarrow (\lambda x)^2+(\lambda y)^2+(\lambda z)^2=4\lambda^2$$

$(\lambda x)^2 + (\lambda y)^2 + (\lambda z)^2 = 4\lambda^2$
$(-3)^2 + (1)^2 + (-1)^2 = 4\lambda^2$
$\lambda = \pm\sqrt{11}/2$

2 cases:
if $\lambda = \sqrt{11}/2$ solving (1,2,3) for x, y, z
obtain point $A = (-6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11})$
$f(A) = 15 - 6(-6/\sqrt{11}) + 2(2/\sqrt{11}) - 2(-2/\sqrt{11})$
$= 15 + 36/\sqrt{11} + 4/\sqrt{11} + 4/\sqrt{11}$
$= 15 + 44/\sqrt{11}$

if $\lambda = -\sqrt{11}/2$ solving (1,2,3) for x, y, z
obtain point $B = (6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11})$
$f(B) = 15 - 6(6/\sqrt{11}) + 2(-2/\sqrt{11}) - 2(2/\sqrt{11})$
$= 15 - 44/\sqrt{11}$

$f(A)$ is max distance^2 & $f(B)$ is min distance^2
b/c $f(A) \perp f(B)$ by Lagrange Multipliers

A is furthest
B is closest

exercise
a box is to be built with surface area 12 what is
max volume of rectangular box without a lid?

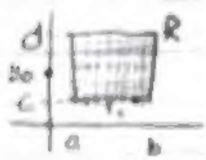
Chapter 15

15.1 Double Integrals

idea: we have functions of several variables. How do we integrate them?

in calc III, a definite integral over region R should be the "net volume of graph of f over R"

$\iint_R f\,dA =$ volume of



$R = [a,b] \times [c,d]$
$= \{(x,y) : x \in (a,b),\ y \in (c,d)\}$

fixing y_0 means working with single variable function $f(x, y_0)$
(can do the same fixing x)
can approximate volume of R by using "left-lower corner points" to determine height of boxes via f(corner point), limiting these approximations yields the true volume...

Fubini's Theorem:
if $f(x,y)$ is continuous on rectangle $R = [a,b] \times [c,d]$, then

$$\int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\,dx\right)dy = \iint_R f(x,y)\,dA = \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)dx$$

example

compute $\iint_R x\sec^2(y)\,dA$ for $R = [0,2] \times [0,\pi/4]$

$$\iint_R x\sec^2(y)\,dA = \int_{y=0}^{\pi/4}\int_{x=0}^{2} x\sec^2(y)\,dx\,dy$$

$$\int_{x=0}^{2} x\sec^2(y)\,dx = \sec^2(y)\int_0^2 x\,dx = \sec^2(y)\left(\frac{x^2}{2}\right)\Big|_0^2$$

$$= \sec^2(y)\frac{1}{2}(4-0) = 2\sec^2(y)$$

$$\int_{y=0}^{\pi/4} 2\sec^2(y)\,dy = 2\tan(y)\Big|_0^{\pi/4} = 2(\tan\pi/4 - \tan 0)$$

$$= 2(1-0) = 2$$

or

$$\iint_R x\sec^2(y)\,dA = \int_{x=0}^{2}\int_{y=0}^{\pi/4} x\sec^2(y)\,dy\,dx$$

$$\int_{y=0}^{\pi/4} x\sec^2(y)\,dy = x(\tan(y))\Big|_0^{\pi/4} = x(\tan\pi/4 - \tan 0)$$

$$= x(1-0) = x$$

$$\int_{x=0}^{2} x\,dx = \frac{x^2}{2}\Big|_0^2 = \frac{4}{2} - \frac{0}{2} = 2$$

example

compute $\iint_R \frac{1}{1+x+y}$ for $R = [1,2] \times [2,3]$

$$\iint_R \frac{1}{1+x+y}\,dA = \int_{x=1}^{2}\int_{y=2}^{3} \frac{1}{1+x+y}\,dy\,dx$$

$$\int_{y=2}^{3} \frac{1}{1+x+y}\,dy \qquad \begin{aligned} u &= 1+x+y \\ du &= dy \end{aligned}$$

$$\int_{y=2}^{3} \frac{1}{u}\,du = \ln|u|\Big|_2^3 = \ln|1+x+y|\Big|_2^3$$

$$= \ln|4+x| - \ln|3+x|$$

$$\int_{x=1}^{2} (\ln|4+x| - \ln|3+x|)\,dx$$

$$= (4+x)(\ln(4+x)-1) - (3+x)(\ln(3+x)-1)\Big|_1^2$$